HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2016

STUDENT NUMBER: ____

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators and drawing templates may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 14

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 The gradient of the tangent to the curve $x^3 - xy^2 + 8 = 0$ at the point $(1,3)$ is:

(A) 1

(B) -1

(C) $\frac{1}{2}$

(D) $\frac{1}{3}$

2 It is known that $x = 2 - 3i$ is a root of $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$. Another root of the equation is:

(A) $x = 1 - 2i$

(B) $x = -1 - 2i$

(C) $x = -2 - i$

(D) $x = -2 + i$

3 The equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is:

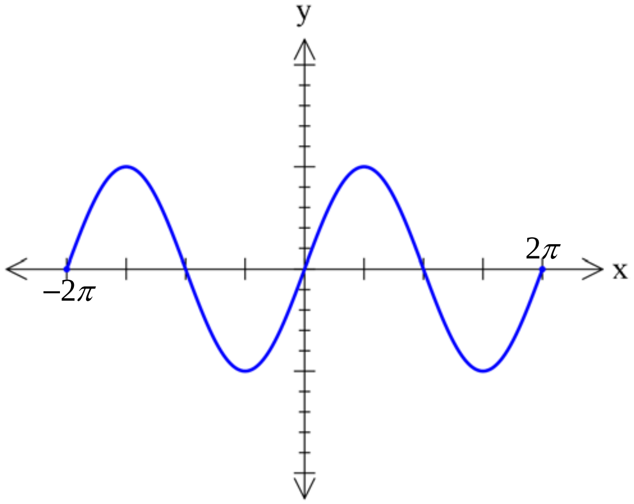
(A) $p^2x - py + c - cp^4 = 0$

(B) $p^3x - py + c - cp^4 = 0$

(C) $x + p^2y - 2c = 0$

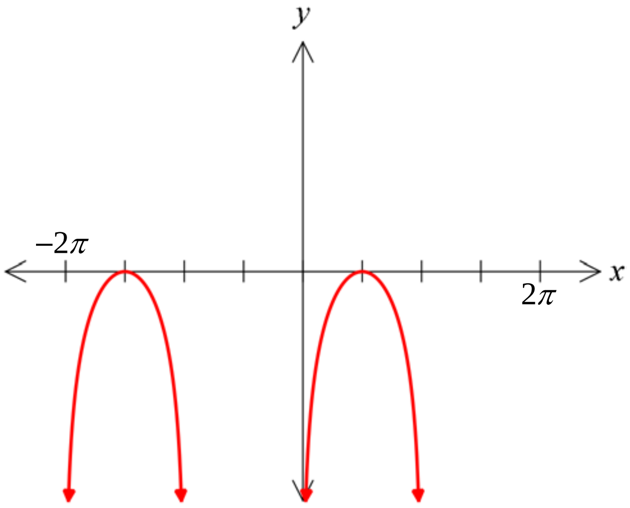
(D) $x + p^2y - 2cp = 0$

4 The graph of $y = \sin x$ for $-2\pi \le x \le 2\pi$ is shown below.

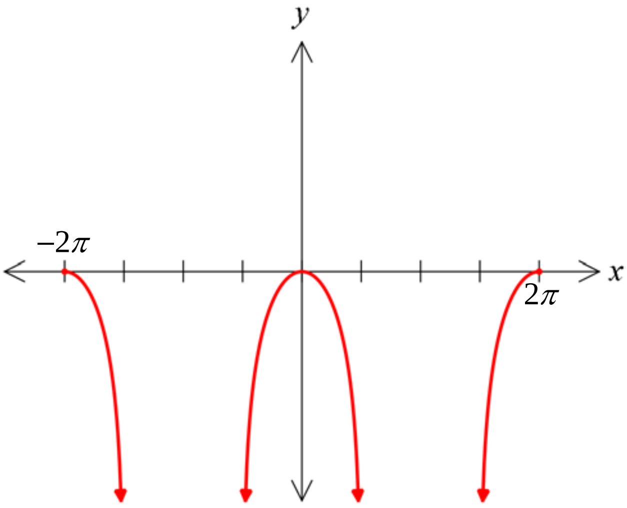


Which of the following is the graph of $y = 2\log_e(\sin x)$ for $-2\pi \le x \le 2\pi$?

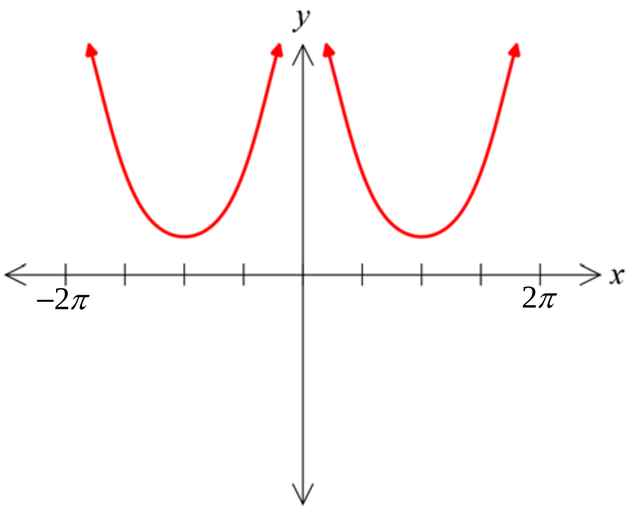
(A)



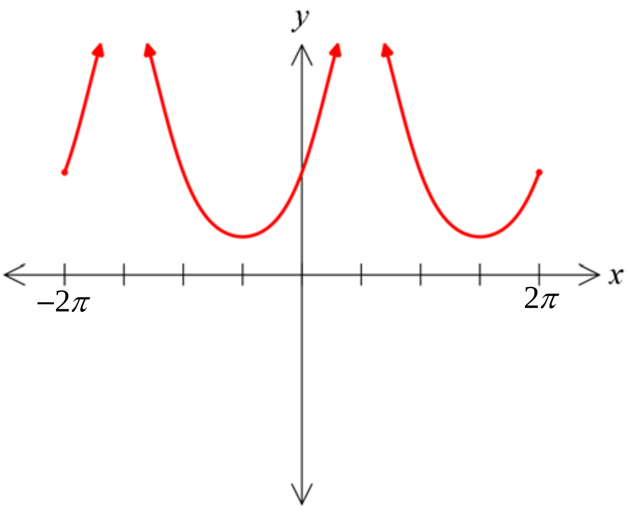
(B)



(C)



(D)



5 If $z = 2 + i$ and $w = 1 - i$, then $z\overline{w}$, in the form $x + iy$ is:

(A) $3 - i$

(B) $1 - 3i$

(C) $1 + 3i$

(D) $1 + i$

6 The reduction formula for $I_n = \int \tan^n x \, dx$ is:

(A) $I_n = \dfrac{\tan^{n+1} x}{n+1} - I_{n-1}$

(B) $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-1}$

(C) $I_n = \dfrac{\tan^{n+1}}{n+1} - I_{n-2}$

(D) $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

7 The roots of the equation $2x^4 - 3x^3 + 4x + 9 = 0$ are α, β, δ and γ.
The value of $\alpha + \beta + \delta + \gamma - \alpha\beta\delta\gamma$ is:

(A) 6

(B) 3

(C) -1

(D) -3

8 Given that $P(x)$ is a polynomial with complex coefficients, it is known that three of the roots of the equation $P(x) = 0$ are $x = 1$, $x = 2 - i$ and $x = 3 + i$.

The minimum degree of $P(x)$ is:

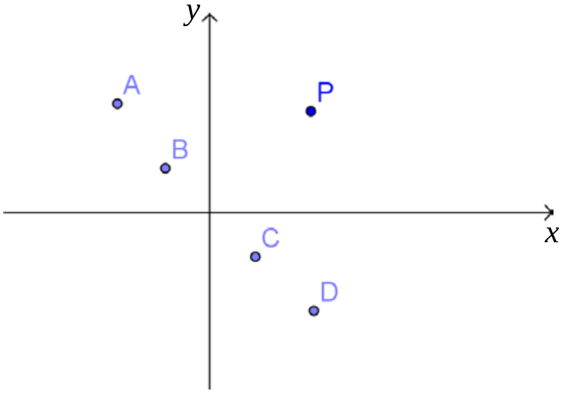
(A) 3

(B) 4

(C) 5

(D) 6

9 The point P on the Argand Diagram below represents a complex number p, where $|p| = \frac{3}{2}$.



The number p^{-1} is best represented by the point

(A) A

(B) B

(C) C

(D) D

10 The equation of the conic with eccentricity $\sqrt{2}$ and asympototes $y = \pm x$ is:

(A) $xy = 2$

(B) $x^2 - y^2 = 4$

(C) $xy = 1$

(D) $\frac{x^2}{4} - y^2 = 1$.

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet

(a) (i) Express $1-i\sqrt{3}$ in modulus-argument form. **1**

(ii) Express $\left(1-i\sqrt{3}\right)^6$ in the form $a+ib$. **2**

(b) (i) Find in modulus-argument form the five roots of $z^5=-1$. **2**

(ii) Prove that when plotted on an Argand diagram these five roots form the vertices of a regular polygon which has an area of approximately 2.38 square units. **2**

(c) Find:

(i) $\int \frac{\tan^{-1} x}{1+x^2}\,dx$ **1**

(ii) $\int \cos^2 x \sin^3 x\,dx$ **2**

(d) The ellipse E has Cartesian equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

(i) Calculate the eccentricity e of the ellipse. **1**

(ii) Find the coordinates of the foci S and S' and the equations of the directricies. **2**

(iii) Sketch the ellipse, showing the above features and intercepts. **2**

Question 12 (15 marks) Start a new writing booklet

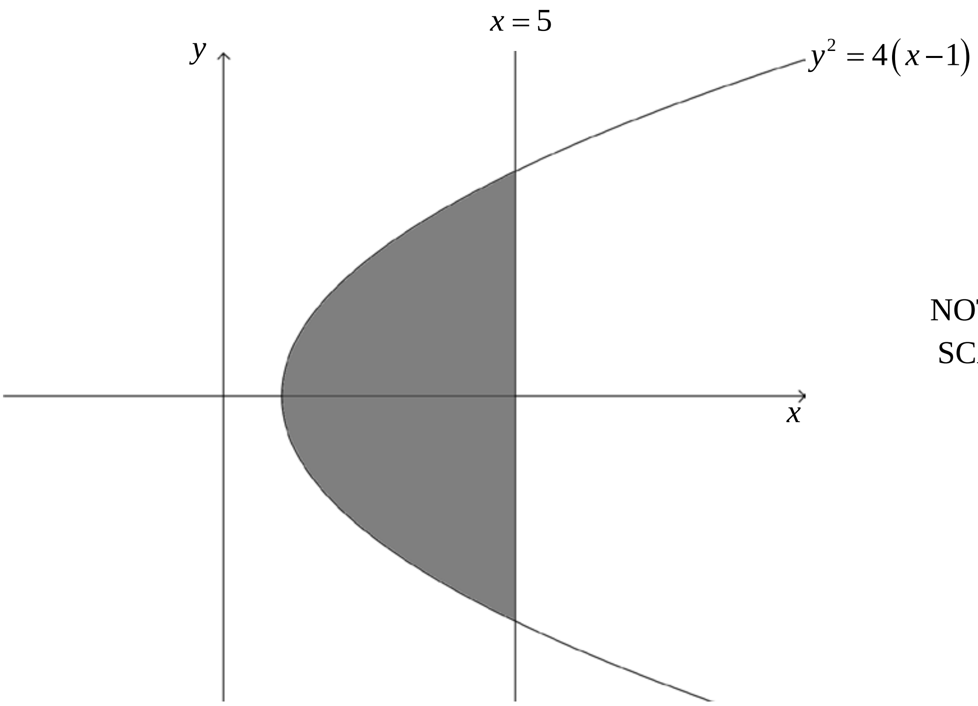
(a) Using the substitution $t = \tan\dfrac{\theta}{2}$, or otherwise, prove that $\displaystyle\int_0^{\frac{\pi}{3}} \frac{d\theta}{2+2\cos\theta} = \frac{\sqrt{3}}{6}$. **3**

(b) The polynomial equation $x^3 - 3x + 4 = 0$ has roots α , β and γ . Find the polynomial equation whose roots are:

(i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. **2**

(ii) $\alpha + 3$, $\beta + 3$ and $\gamma + 3$ **2**

(c) The diagram shows the region bounded by $y^2 = 4(x-1)$ and the line $x = 5$. **3**



NOT TO SCALE

By using cylindrical shells, or otherwise, find the volume of the solid formed by rotating the given region about the y -axis.

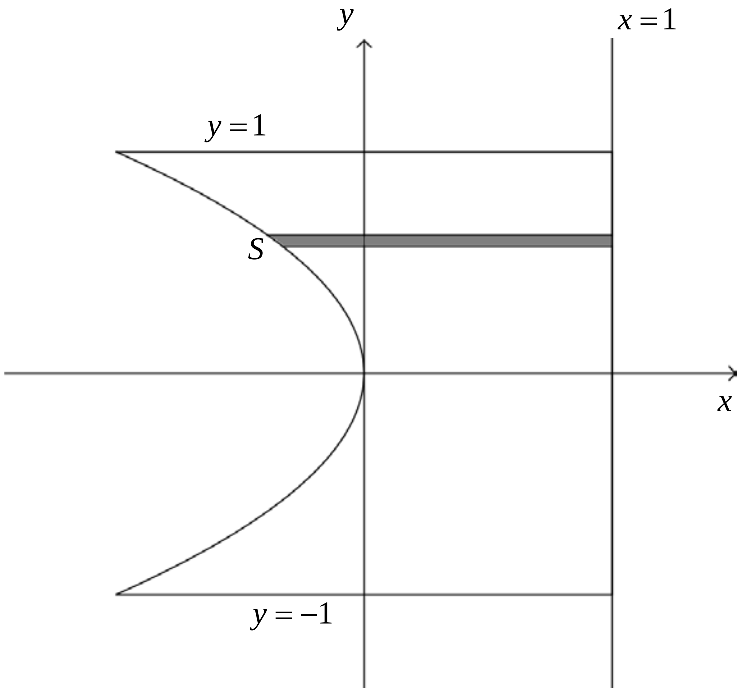
(d) (i) If $I_n = \int_0^1 x^n (1-x)^{\frac{1}{2}}\, dx$, show that for $n > 0$, $I_n = \dfrac{2n}{2n+3} I_{n-1}$ where n is an integer. **3**

(ii) Hence, or otherwise, evaluate $\int_0^1 x^3 (1-x)^{\frac{1}{2}}\, dx$. **2**

Question 13 (15 marks) Start a new writing booklet

(a) On the Argand diagram, shade the region specified by both the conditions $\text{Re}(z) \leq 4$ and $|z - 4 + 5i| \leq 3$, showing the points of intersection of the boundaries. **2**

(b) Consider the region bounded by the lines $x = 1$, $y = 1$ and $y = -1$ and by the curve $x + y^2 = 0$. The region is rotated through $360°$ about the line $x = 4$ to form a solid. When the region is rotated, the line segment at the point $S(x, y)$ on $x + y^2 = 0$ sweeps out an annulus.



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(i) Show that the area of the annulus at height y is equal to $\pi(y^4 + 8y^2 + 7)$. **2**

(ii) Hence find the volume of the solid. **2**

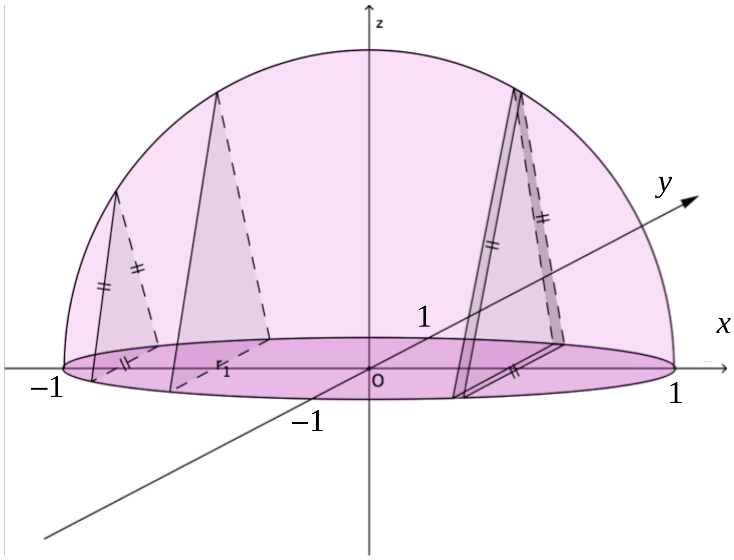
(c) (i) If a, b, c are real and unequal show that $a^2 + b^2 > 2ab$. **1**

(ii) Hence deduce that $a^2 + b^2 + c^2 > ab + bc + ca$. **2**

Question 13 continues on page 10

Question 13 (continued)

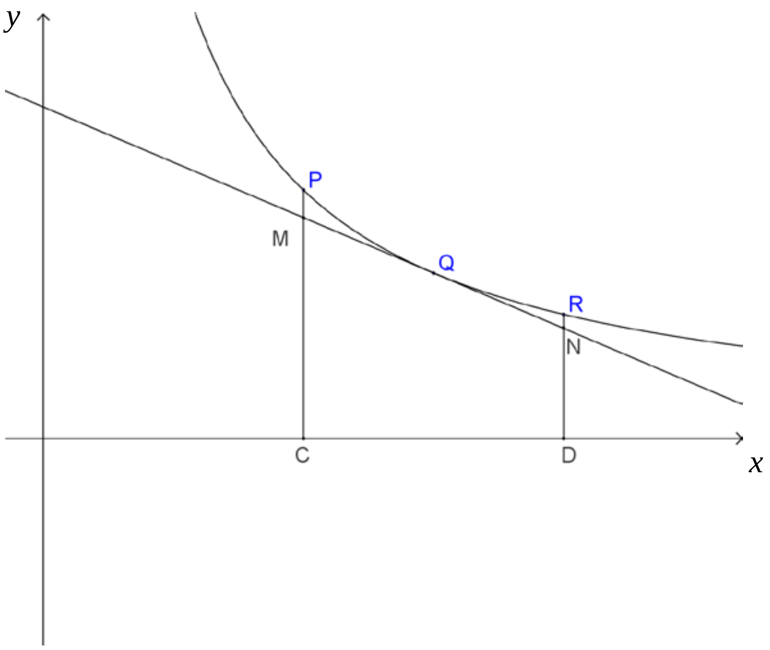
(d) The diagram below shows a solid with circular base of radius 1. Parallel cross sections perpendicular to the base and the x-axis are equilateral triangles. Find the volume of the solid. **3**



NOT TO SCALE

(e) The points P, Q and R on the curve $y=\dfrac{2}{x}$ have x-coordinates 1, $\dfrac{3}{2}$ and 2 respectively.

The points C and D are the feet of the perpendiculars drawn from P and R to the x-axis. The tangent to the curve at Q with equation $8x+9y-24=0$ cuts PC and RD at M and N respectively.



NOT TO SCALE

(i) Find the coordinates of M and N. **1**

(ii) Using areas and integration, show that $\dfrac{2}{3}<\ln 2<\dfrac{3}{4}$. **2**

End of Question 13

Question 14 (15 marks) Start a new writing booklet

(a) (i) If $\dfrac{16x}{x^4-16} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx}{x^2+4}$, find the values of A, B and C. **2**

(ii) Hence, show that $\displaystyle\int_4^6 \frac{16x}{x^4-16}\,dx = \log_e\left(\frac{4}{3}\right)$. **2**

(b) Consider the function $f(x) = \dfrac{(x-2)(x+1)}{x-4}$ for $x \neq 4$.

(i) Show that $f(x) = x + 3 + \dfrac{10}{x-4}$. **1**

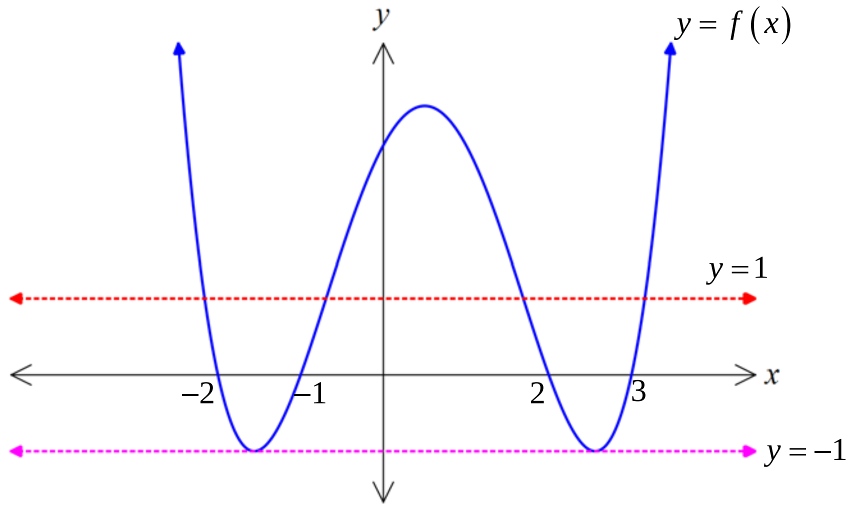
(ii) Explain why the graph of $y = f(x)$ approaches $y = x+3$ as x approaches ∞. **1**

(iii) Find the values of x for which $f(x)$ is positive. **2**

(iv) Show that the graph of $y = f(x)$ has two stationary points. **1**
You do not need to find the y values of the stationary points.

(v) Sketch the curve $y = f(x)$, labelling all asymptotes and x intercepts. **2**

(c) The graphs of $y = f(x)$, $y = 1$ and $y = -1$ are shown below.



Sketch each of the following on a separate number plane:

(i) $y = |f(x)|$ **1**

(ii) $y^2 = f(x)$ **2**

(iii) $y = f(|x|)$ **1**

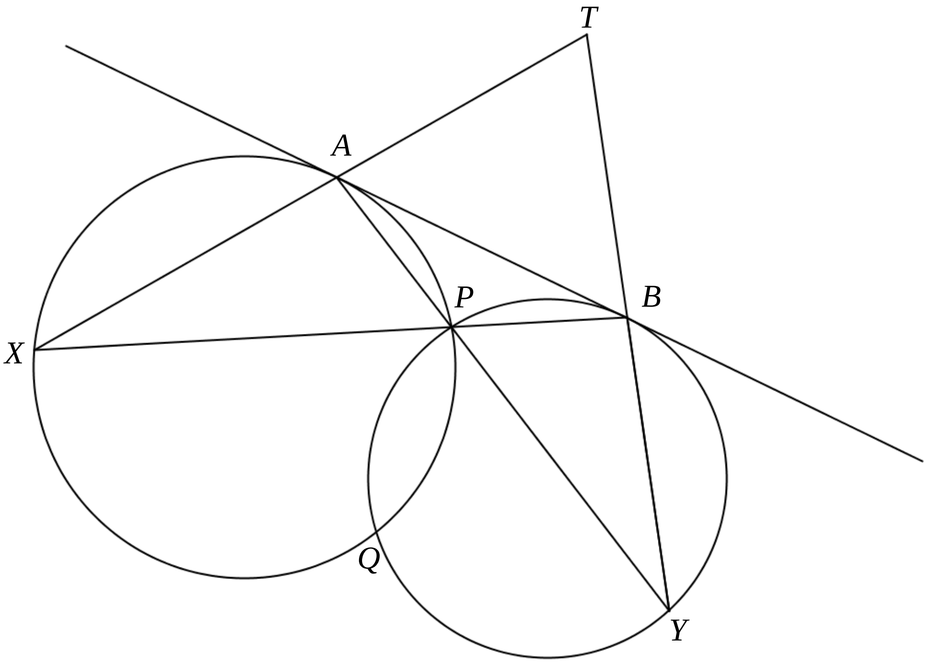
Question 15 (15 marks) Start a new writing booklet

(a) (i) Use the substitution $x = a - y$ to show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **1**

(ii) Hence, or otherwise, evaluate $\int_0^1 x(1-x)^{20}\,dx$. **2**

(b) In the diagram below, AB is a common tangent of the two circles which intersect at P and Q.

XPB and APY are straight lines. XA produced and YB produced meet at T.



NOT TO SCALE

Copy or trace the diagram into your writing booklet

(i) Prove that $AT = TB$. **1**

(ii) If $ATBP$ is a cyclic quadrilateral, find the size of $\angle TAB$. **2**

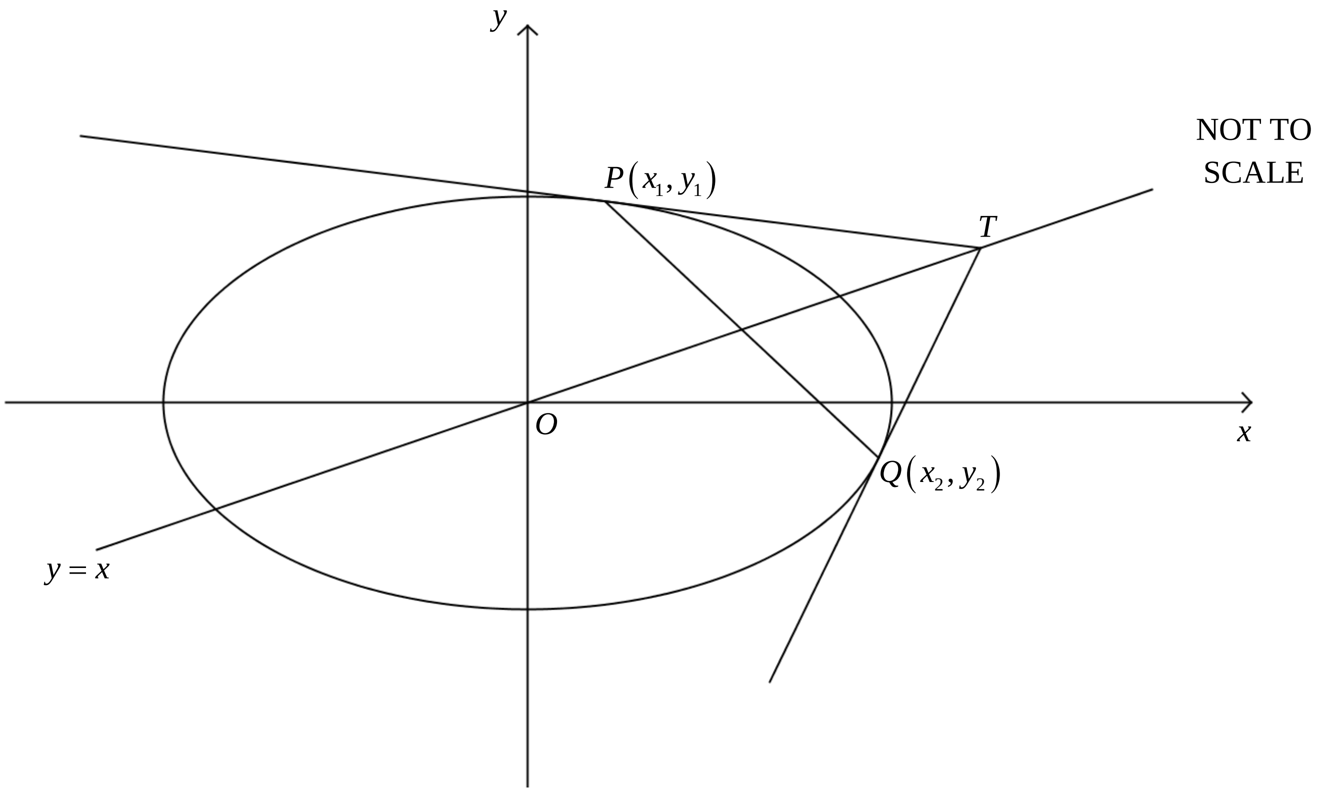
(c) (i) Use the substitution $u = -x$ to show that $\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx = \int_{-2}^{2} \frac{x^2 e^x\,dx}{e^x+1}$. **1**

(ii) Hence, or otherwise, evaluate $\int_{-2}^{2} \frac{x^2}{e^x+1}\,dx$. **2**

Question 15 continues on page 13

Question 15 (continued)

(d) The line $y = x$ meets a directrix of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a > b$ at the point T in the first quadrant. Tangents from T meets the ellipse at $P(x_1, y_1)$ and $Q(x_2, y_2)$. The eccentricity of the ellipse is e.



(i) Given that the chord of contact of the tangents from the point (x_0, y_0) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has equation $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$, deduce that the equation of PQ is $\frac{x}{ae}+\frac{y}{ae\left(1-e^2\right)}=1$ and verify that PQ is a focal chord of the ellipse. **2**

(ii) Show that x_1 and x_2 are roots of the equation **2**

$\left(2-e^2\right)x^2-2ae\left(1-e^2\right)x+a^2\left(e^2-e^4-1\right)=0$.

(iii) Show that the midpoint M of the chord PQ lines on the line $y = x$. **2**

End of Question 15

Question 16 (15 marks) Start a new writing booklet

(a) z is a complex number which satisfies $|z-1|=1$. Let $\arg(z)=\theta$, where θ is acute.

(i) Show graphically that $\arg(z-1)=2\theta$. **1**

(ii) Hence, or otherwise, find $\arg\left(z^2-3z+2\right)$ in terms of θ. **2**

(b) If $a>0$, $b>0$ and $c>0$, and $a+\frac{1}{a}\geq 2$, show that:

(i) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$ **1**

(ii) $(a+b)(b+c)(c+a)\geq 8abc$ **2**

(c) A particle of mass m is projected vertically upwards under gravity. The air resistance to the motion is $\frac{1}{100}mgv^2$ where v is the speed of the particle.

(i) Show that during the upward motion of the particle, if x is the upward vertical displacement of the particle from its projection point at time t, then $\ddot{x}=-\frac{1}{100}g\left(100+v^2\right)$. **1**

(ii) If the speed of projection is u, show that the greatest height (above the projection point) reached by the particle is $\frac{50}{g}\log_e\left(\frac{100+u^2}{100}\right)$. **2**

(iii) Show that during the downward motion of the particle, if x is the downward vertical displacement of the particle from its highest position at time t after it begins the downward motion, then $\ddot{x}=\frac{1}{100}g\left(100-v^2\right)$. **2**

(iv) Show that the speed of the particle on return to its point of projection is $\frac{10u}{\sqrt{100+u^2}}$. **2**

(v) Find the terminal velocity V of the particle for the downward motion. **1**

(vi) If the initial speed of projection of the particle is V, show that the speed on return to the point of projection is $\frac{1}{\sqrt{2}}V$. **1**

End of Examination

Mathematics Extension 2 Trial 2016
Multiple Choice
Question 1
$x^3 - xy^2 + 8 = 0$
Implicitly differentiating:
$$3x^2 - y^2 - 2xy\frac{dy}{dx} = 0$$
$$2xy\frac{dy}{dx} = 3x^2 - y^2$$
$$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy}$$
At $(1,3)$
$$\frac{dy}{dx} = \frac{3-9}{2\times1\times3}$$
$$= \frac{-6}{6}$$
$$= -1$$
(B)

Question 2:

Σ roots one at a time $= 6$
$2+2+a+a = 6$
$a = 1$
$(2-3i)+(2+3i)+(a+ib)+(a-ib)=6$
$4+2a = 6$
$2a = 2$
$a = 1$ $\therefore$ (A)

Question 3:
$xy = c^2$
$y = c^2x^{-1}$
$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$
At $x = cp$
$$\frac{dy}{dx} = \frac{-c^2}{c^2p^2}$$
$$= \frac{-1}{p^2}$$
Gradient of normal is p^2

$$y-\frac{c}{p}=p^2\left(x-cp\right)$$
$$py-c=p^3x-cp^4$$
$$p^3x-py+c-cp^4=0$$
(B)

Question 4:
(A)

Question 5
$$z\overline{w}=\left(2+i\right)\left(1+i\right)$$
$$=2+2i+i-i^2$$
$$=1+3i$$
(C)
Question 6
$$I_n=\int\tan^n xdx$$
$$=\int\tan^2 x\tan^{n-2} xdx$$
$$=\int\sec^2 x\tan^{n-2} xdx-\int\tan^{n-2} xdx$$
$$=\frac{\tan^{n-1} x}{n-1}-I_{n-2}$$
(D)

Question 7
$$\alpha+\beta+\gamma+\delta=\frac{3}{2}$$
$$\alpha\beta\gamma\delta=\frac{9}{2}$$
$$\frac{3}{2}-\frac{9}{2}=-3$$
(D)

Question 8:
Three known roots, therefore degree 3 as polynomial may have unreal coefficients, so not essential they occur in conjugate pairs.
(A)

Question 9:
$$|\,p\,|=\frac{3}{2}$$
$$\left|\frac{1}{p}\right|=\frac{2}{3}$$
$$\arg(p)>0$$
$$\arg(p^{-1})<0$$
Therefore (C)

Question 10:

Due to asymptotes, know it is not hyperbola in form $xy = c^2$

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$2 = 1 + \frac{b^2}{a^2}$$

$$\therefore b = a$$

(B)

Question 11

(a)

Let $z = 1 - i\sqrt{3}$

(i)

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{4}$$
$$= 2$$

$$\arg(z) = -\tan^{-1}\frac{\sqrt{3}}{1}$$
$$= -\frac{\pi}{3}$$

(ii)

$$\left(1 - i\sqrt{3}\right)^6 = 2^6\left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right)$$
$$= 2^6\left(\cos 2\pi + i\sin 2\pi\right)$$
$$= 2^6$$
$$= 64$$

(b)

(i)

$$z^5 = -1$$
$$= \cos\pi + i\sin\pi$$
$$z = \cos\left(\frac{2k\pi + \pi}{5}\right) + i\sin\left(\frac{2k\pi + \pi}{5}\right)$$

Let $k = 0$:

$$z_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

Let $k = 1$:

$$z_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$

Let $k = 2$:

$$z_2 = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}$$
$$= \cos\pi + i\sin\pi$$

Let $k = 3$:

$$z_3 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$
$$= \cos\left(-\frac{3\pi}{5}\right) + i\sin\left(-\frac{3\pi}{5}\right)$$

Let $k = 4$:

$$z_4 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$
$$= \cos\left(-\frac{\pi}{5}\right) + i\sin\left(-\frac{\pi}{5}\right)$$

(ii)

$$Area = 5\times\frac{1}{2}\times a\times b\times\sin\theta$$
$$= 5\times1\times1\times\sin\frac{2\pi}{5}$$
$$= 2.3776...$$
$$= 2.38\text{ units}^2 \quad (2dp)$$

(c)
(i)

$$\int\frac{\tan^{-1}x}{1+x^2}dx = \int\frac{1}{1+x^2}\tan^{-1}xdx$$
$$= \frac{\left(\tan^{-1}x\right)^2}{2}+C$$

(ii)

$$\int\cos^2 x\sin^3 xdx = \int\cos^2 x\left(1-\cos^2 x\right)\sin xdx$$
$$= \int\cos^2 x\sin x-\cos^4 x\sin xdx$$
$$= \frac{-\cos^3 x}{3}+\frac{\cos^5 x}{5}+C$$

(d)

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

(i)

$$a=2, b=3$$
$$4=9\left(1-e^2\right)$$
$$\frac{4}{9}=1-e^2$$
$$e^2=\frac{5}{9}$$
$$e=\frac{\sqrt{5}}{3} \quad (\text{e}>0)$$

(ii)

$$S=\left(0,be\right)$$
$$=\left(0,3\times\frac{\sqrt{5}}{3}\right)$$
$$=\left(0,\sqrt{5}\right)$$
$$S'=\left(0,-\sqrt{5}\right)$$

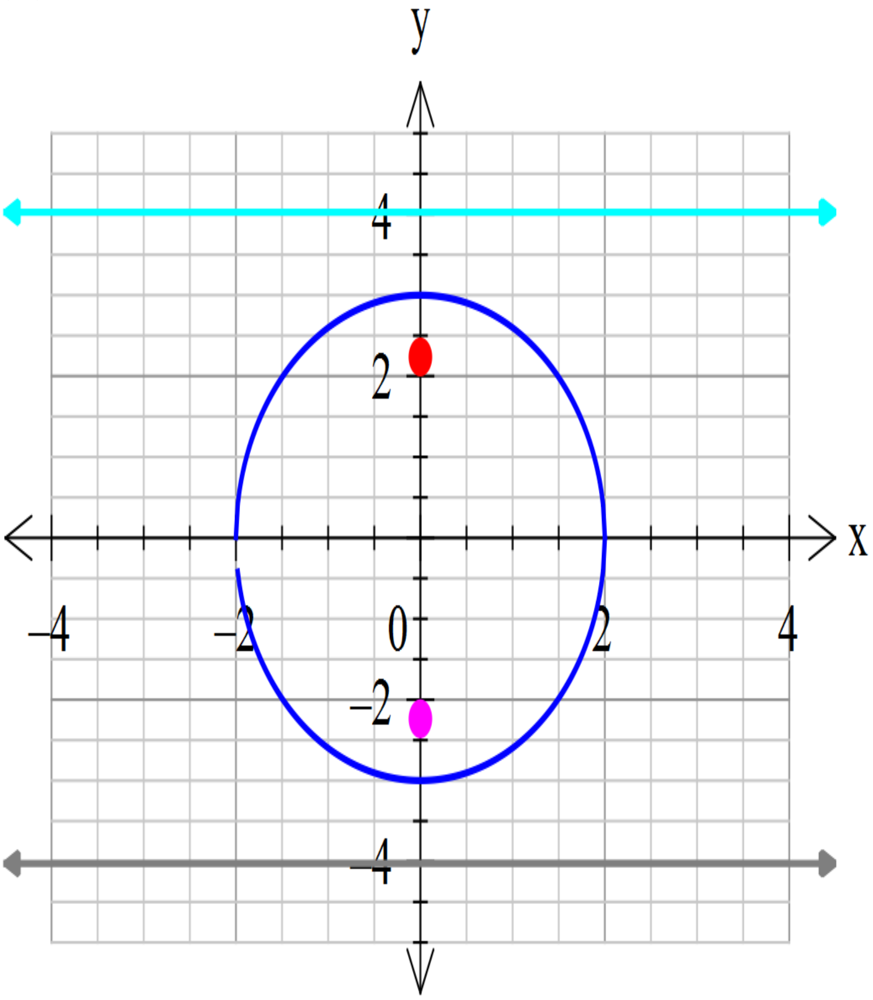
Directricies:

$$y = \pm \frac{b}{e}$$

$$= \pm 3 \times \frac{3}{\sqrt{5}}$$

$$= \pm \frac{9}{\sqrt{5}}$$

(iii)



Question 12

(a)

$$Let\ I=\int_0^{\frac{\pi}{3}}\frac{d\theta}{2+2\cos\theta}$$

$$Let\ t=\tan\frac{\theta}{2}$$

$$\theta=2\tan^{-1}t$$

$$\frac{d\theta}{dt}=\frac{2}{1+t^2}$$

$$d\theta=\frac{2dt}{1+t^2}$$

When $\theta=0, t=0$ and when $\theta=\frac{\pi}{3}, t=\frac{1}{\sqrt{3}}$

$$\therefore I=\int_0^{\frac{1}{\sqrt{3}}}\left(\frac{2dt}{1+t^2}\right)\div\left(2+2\left(\frac{1-t^2}{1+t^2}\right)\right)$$

$$=\int_0^{\frac{1}{\sqrt{3}}}\left(\frac{2dt}{1+t^2}\right)\div\left(\frac{2+2t^2+2+2t^2}{1+t^2}\right)$$

$$=\frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}}dt$$

$$=\frac{1}{2}[t]_0^{\frac{1}{\sqrt{3}}}$$

$$=\frac{1}{2}\times\frac{1}{\sqrt{3}}$$

$$=\frac{1}{2}\times\frac{\sqrt{3}}{3}$$

$$=\frac{\sqrt{3}}{6}$$

(b)

$x^3-3x+4=0$

(i)

$$\left(\frac{1}{x}\right)^3-3\left(\frac{1}{x}\right)+4=0$$

$$\frac{1}{x^3}-\frac{3}{x}+4=0$$

$$1-3x^2+4x^3=0$$

$$4x^3-3x^2+1=0$$

(ii)

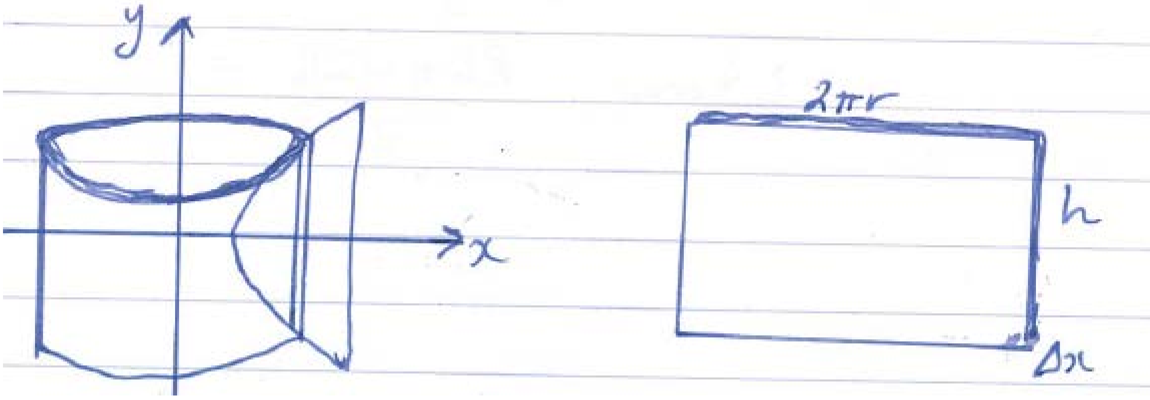
$$(x-3)^3-3(x-3)+4=0$$

$$x^3-3x^2\times3+3x\times3^2-27-3x+9+4=0$$

$$x^3-9x^2+27x-27-3x+13=0$$

$$x^3-9x^2+24x-14=0$$

(c)



$h = 2y$

$r = x$

$$\Delta V = 2\pi rh$$

$$= 2\pi x \times 2y\Delta x$$

$$= 4\pi x \times 2\sqrt{x-1}\Delta x$$

$$= 8\pi x\sqrt{x-1}\Delta x$$

$$V \approx \sum_{x=1}^{5} 8\pi x\sqrt{x-1}\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=1}^{5} 8\pi x\sqrt{x-1}\Delta x$$

$$= 8\pi \int_1^5 x\sqrt{1-x}dx$$

$$= 8\pi \int_0^4 (u+1)\sqrt{u}\, du$$

$$= 8\pi \int_0^4 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du$$

$$= 8\pi \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} \right]_0^4$$

$$= 8\pi \left(\frac{2}{5} \times 4^{\frac{5}{2}} + \frac{2}{3} \times 4^{\frac{3}{2}} \right)$$

$$= 8\pi \times \frac{272}{15}$$

$$= \frac{2176}{15}\pi \text{ units}^3$$

(d)
(i)

Let $I_n = \int_0^1 x^n (1-x)^{\frac{1}{2}} dx$

$u = x^n$ $\qquad$ $v' = (1-x)^{\frac{1}{2}}$

$u = nx^{n-1}$ $\qquad$ $v = \frac{-2}{3}(1-x)^{\frac{3}{2}}$

$$I_n = \left[\frac{-2}{3}x^n(1-x)^{\frac{3}{2}}\right]_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)(1-x)^{\frac{1}{2}}\,dx$$

$$= 0 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)^{\frac{1}{2}}\,dx - \frac{2n}{3}\int_0^1 x^n(1-x)^{\frac{1}{2}}\,dx$$

$$= \frac{2n}{3}I_{n-1} - \frac{2n}{3}I_n$$

$$I_n\left(1+\frac{2n}{3}\right) = \frac{2n}{3}I_{n-1}$$

$$I_n\left(\frac{3+2n}{3}\right) = \frac{2n}{3}I_{n-1}$$

$$I_n = \frac{2n}{3+2n}I_{n-1}$$

(ii)

$$I_3 = \frac{6}{9}I_2$$

$$= \frac{6}{9}\times\frac{4}{7}I_1$$

$$= \frac{6}{9}\times\frac{4}{7}\times\frac{2}{5}I_0$$

$$= \frac{16}{105}\int_0^1 (1-x)^{\frac{1}{2}}dx$$

$$= \frac{16}{105}\left[\frac{-2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = \frac{16}{105}\left(\frac{2}{3}\right)$$

$$= \frac{32}{315}$$

Question 13

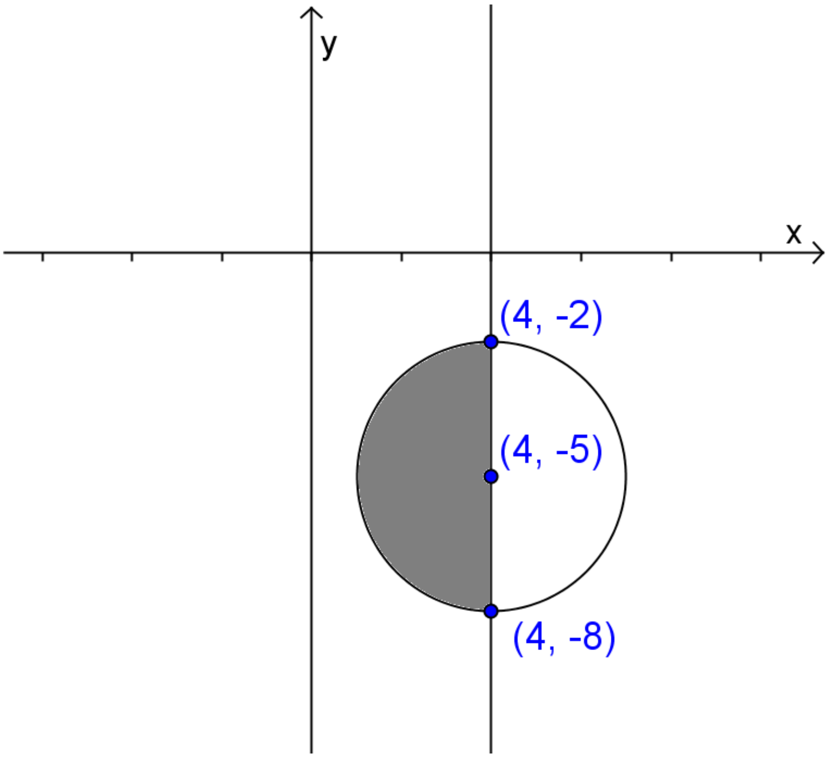
(a)

$\text{Re}(z) \leq 4$

$x \leq 4$

$|z-(4-5i)| \leq 3$

Circle with centre $(4,-5)$ and radius is 3.



(b)

(i)

$A = \pi(R^2 - r^2)$

$= \pi\left[(4-x)^2 - 3^2\right]$

$= \pi\left[(4+y^2) - 9\right]$

$= \pi(16 + 8y^2 + y^4 - 9)$

$= \pi(y^4 + 8y^2 + 7)$

(ii)

$$V \approx \sum_{y=-1}^{1} \pi\left(y^4 + 8y^2 + 7\right)\Delta y$$

$$V = \lim_{\Delta y \to 0} \sum_{y=-1}^{1} \pi\left(y^4 + 8y^2 + 7\right)\Delta y$$

$$= \pi \int_{-1}^{1} \left(y^4 + 8y^2 + 7\right) dy$$

$$= 2\pi \int_{0}^{1} \left(y^4 + 8y^2 + 7\right) dy$$

$$= 2\pi \left[\frac{y^5}{5} + \frac{8y^3}{3} + 7y\right]_0^1$$

$$= 2\pi \left[\frac{1}{5} + \frac{8}{3} + 7\right]$$

$$= \frac{296\pi}{15}$$

(c)
(i)

$\left(a-b\right)^2 > 0$ since $a - b \neq 0$

$$a^2 - 2ab + b^2 > 0$$

$$a^2 + b^2 > 2ab$$

(ii)

$$a^2 + b^2 > 2ab$$

$$b^2 + c^2 > 2bc$$

$$c^2 + a^2 > 2ac$$

$$\therefore a^2 + b^2 + b^2 + c^2 + c^2 + a^2 > 2ab + 2bc + 2ac$$

$$2a^2 + 2b^2 + 2c^2 > 2\left(ab + bc + ac\right)$$

$$2\left(a^2 + b^2 + c^2\right) > 2\left(ab + bc + ac\right)$$

$$a^2 + b^2 + c^2 > ab + bc + ac$$

(ii)

$$\Delta V = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 2y \times 2y \times \sin 60° \Delta x$$

$$= 2y^2 \times \frac{\sqrt{3}}{2} \Delta x$$

$$= \sqrt{3} y^2 \Delta x$$

$$= \sqrt{3}\left(1 - x^2\right)\Delta x$$

$$V \approx \sum_{-1}^{1} \sqrt{3}\left(1-x^2\right)\Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{-1}^{1} \sqrt{3}\left(1-x^2\right)\Delta x$$

$$= \int_{-1}^{1} \sqrt{3}\left(1-x^2\right)dx$$

$$= 2\int_{0}^{1} \sqrt{3}\left(1-x^2\right)dx$$

$$= 2\sqrt{3}\left[x-\frac{x^3}{3}\right]_0^1$$

$$= 2\sqrt{3}\left(1-\frac{1}{3}\right)$$

$$= \frac{4\sqrt{3}}{3} \textit{ cubic units}$$

(e)

(i)

$8x+9y-24=0$

Let $x=1$

$8+9y-24=0$

$9y=16$

$y=\frac{16}{9}$

$M\left(1,\frac{16}{9}\right)$

Let $x=2$

$16+9y-24=0$

$9y=8$

$y=\frac{8}{9}$

$N\left(2,\frac{8}{9}\right)$

(ii)

$$A_{MNCD} < \int_1^2 \frac{2}{x}dx < A_{PRDC}$$

$$\frac{1}{2}\left(\frac{16}{9}+\frac{8}{9}\right) < 2\left[\ln x\right]_1^2 < \frac{1}{2}(2+1)$$

$$\frac{4}{3} < 2\ln 2 - 2\ln 1 < \frac{3}{2}$$

$$\frac{2}{3} < \ln 2 < \frac{3}{4}$$

Question 14

(a)

(i)

$$\frac{16x}{x^4-16} \equiv \frac{A}{x-2}+\frac{B}{x+2}+\frac{Cx}{x^2+4}$$

$$16x = A(x+2)(x^2+4)+B(x-2)(x^2+4)+Cx(x^2-4)$$

$$Let\ x=2$$

$$32 = A\times 4\times 8$$

$$A=1$$

$$Let\ x=-1$$

$$-16 = B(-4)(8)$$

$$B=1$$

$$Let\ x=1$$

$$16=(3)(5)+(-1)(5)+C(-3)$$

$$16=15-5-3C$$

$$6=-3C$$

$$C=-2$$

(ii)

$$\int_4^6 \frac{16x}{x^4-16}dx = \int_4^6\left(\frac{1}{x-2}+\frac{1}{x+2}-\frac{2x}{x^2+4}\right)dx$$

$$=\left[\ln|x-2|+\ln|x+2|-\ln|x^2+4|\right]_4^6$$

$$=\left[\ln\frac{x^2-4}{x^2+4}\right]_4^6$$

$$=\frac{\ln 32}{40}-\ln\frac{12}{20}$$

$$=\ln\frac{4}{5}+\ln\frac{5}{3}$$

$$=\ln\left(\frac{4}{5}\times\frac{5}{3}\right)$$

$$=\ln\frac{4}{3}$$

(b)

$$f(x)=\frac{(x-2)(x+1)}{x-3}$$

(i)

$$x+3+\frac{10}{x-4}=\frac{(x+3)(x-4)+10}{x-4}$$
$$=\frac{x^2+3x-4x-12+10}{x-4}$$
$$=\frac{x^2-x-2}{x-4}$$
$$=\frac{(x-2)(x+1)}{x-4}$$

(ii)

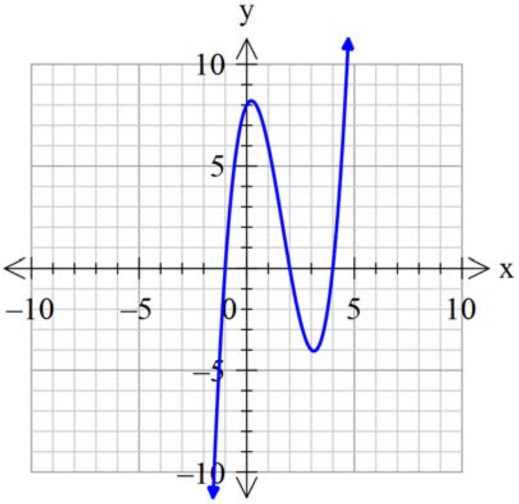
As $x\to\infty$, $\frac{10}{x-4}\to o$

$\therefore y\to x+3$

(iii) Let $f(x)>0$

$$\frac{(x-2)(x+1)}{x-4}>0$$

$(x-4)(x-2)(x+1)>0$



$\therefore f(x)>0 \ for \ x>4, -1<x<2$

(iv)

$f(x)=x+3+10(x-4)^{-1}$

$$f'(x)=1-\frac{10}{(x-4)^2}$$

$Let \ f'(x)=0$

$$0=1-\frac{10}{(x-4)^2}$$

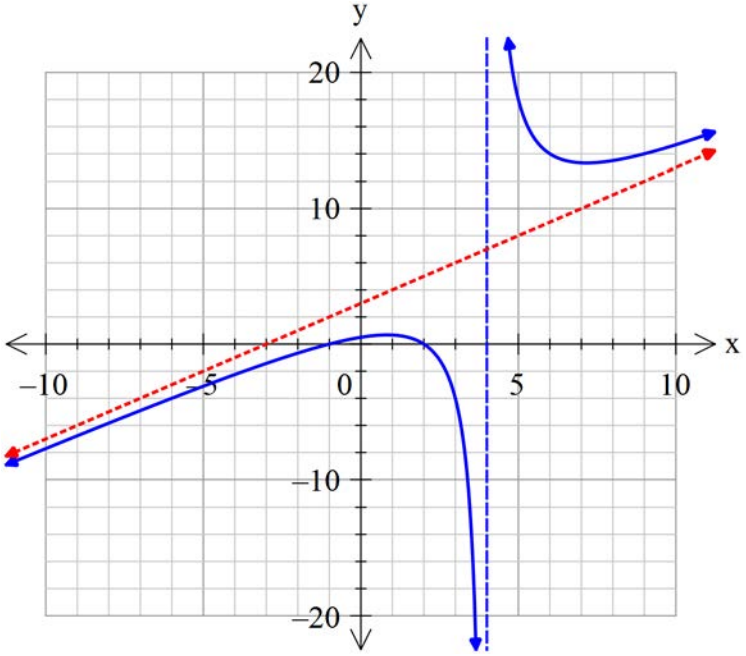
$0=(x-4)^2-10$

$(x-4)^2=10$

$x=4\pm\sqrt{10}$

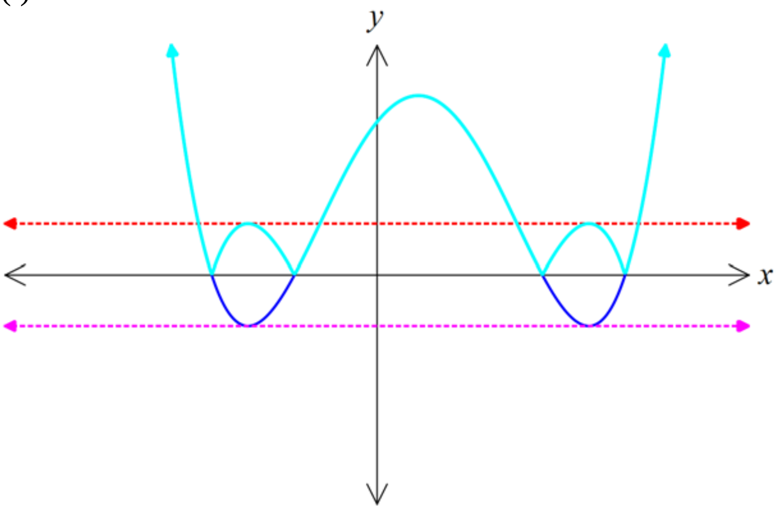
Therefore there are two stationary points.

(v)

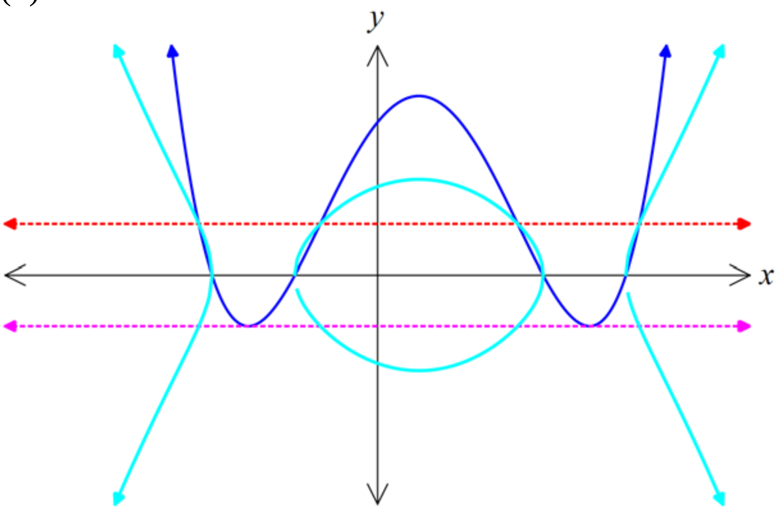


(b)

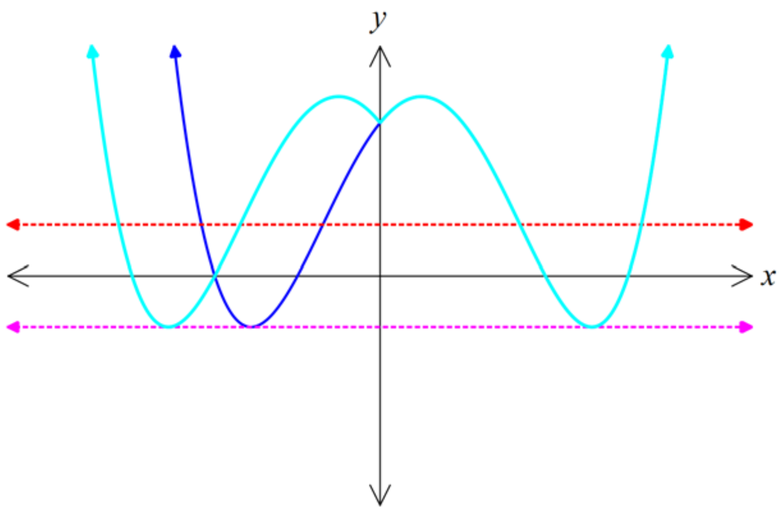
(i)



(ii)



(iii)


y
x

Question 15
(a)
(i)
$x = a - y$

$dx = -dy$

$When\ x = 0,\ y = a$

$When\ x = 1,\ y = 0$

$$\int_0^a f(x)\,dx = -\int_a^0 f(a-y)\,dy$$

$$= \int_0^a f(a-y)\,dy$$

$$= \int_0^a f(x-a)\,dx$$

(ii)

$$f(x) = x(1-x)^{20}$$

$$f(1-x) = (1-x)(1-(1-x))^{20}$$

$$= (1-x)x^{20}$$

$$\int_0^1 x(1-x)^{20}\,dx = \int_0^1 (1-x)x^{20}dx$$

$$= \int_0^1 \left(x^{20} - x^{21}\right)dx$$

$$= \left[\frac{x^{21}}{21} - \frac{x^{22}}{22}\right]_0^1$$

$$= \frac{1}{21} - \frac{1}{22}$$

$$= \frac{1}{462}$$

(b)
(i)
Let $\angle TAB = \alpha$
$\angle MAX = \alpha$ (vertically opposite angles)
$\angle APX = \alpha$ (angle in the alternate segment)
$\angle BPT = \alpha$ (vertically opposite angles)
$\angle NBH = \alpha$ (angle in the alternate segment theorem)
$\angle TBA = \alpha$ (vertically opposite angles)
$\therefore \angle TAB = \angle TBA$
$\therefore AT = BT$ (equal sides opposite equal angles in an isosceles triangle)

(ii)
$\angle BPA + \alpha = 180°$ (angles on the straight line)
$\angle BPA = 180° - \alpha$
$\angle ATB + 2\alpha = 180°$ (angle sum of a triangle)
$\angle ATB = 180° - \alpha$
$\angle BPA + \angle ATB = 180°$ (opposite angles of a cyclic quadrilateral are equal).

$180° - \alpha + 180° - 2\alpha = 180°$

$360° - 3\alpha = 180°$

$3\alpha = 180°$

$\alpha = 60°$

$\therefore \angle TAB = 60°$

(c)
(i)
$u = -x$

$du = -dx$

When $x = -2$, $u = 2$
When $x = 2$, $u = -2$

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx = -\int_{2}^{-2} \frac{u^2}{e^{-u} + 1} du$$

$$= \int_{-2}^{2} \frac{u^2 e^u}{1 + e^u} du$$

$$= \int_{-2}^{2} \frac{x^2 e^x}{1 + e^x} dx$$

(ii)

Let $I = \int_{-2}^{2} \frac{x^2}{e^x + 1} dx$

$$2I = \int_{-2}^{2} \frac{x^2}{e^x + 1} dx + \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx$$

$$= \int_{-2}^{2} \frac{x^2 + x^2 e^x}{e^x + 1} dx$$

$$= \int_{-2}^{2} x^2 dx$$

$$= 2\int_{0}^{2} x^2 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^2$$

$$= 2 \times \frac{8}{3}$$

$$= \frac{16}{3}$$

$$\therefore I = \frac{8}{3}$$

(d)
(i)
$T\left(\frac{a}{e}, \frac{a}{e}\right)$ as it lies on the line $y = x$ and the directrix $x = \frac{a}{e}$.

$\therefore PQ$ has equation:

$$\frac{x}{a^2}\times\frac{a}{e}+\frac{y}{b^2}\times\frac{a}{e}=1$$

$$\frac{x}{ae}+\frac{ay}{e}\times\frac{1}{a^2(1-e)^2}=1 \text{ since } b^2=a^2(1-e^2)$$

$$\frac{x}{ae}+\frac{y}{ae(1-e)^2}=1$$

Sub $S(ae,0)$

$$LHS=\frac{ae}{ae}+0$$
$$=1$$
$$=RHS$$

Therefore PQ is a focal chord.

(ii)

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \quad ... (1)$$

$$\frac{x}{ae}+\frac{y}{ae(1-e^2)}=1 \quad ... (2)$$

From (2)

$$\frac{y}{ae(1-e^2)}=1-\frac{x}{ae}$$

$$\frac{y}{ae(1-e^2)}=\frac{ae-x}{ae}$$

$$y=(ae-x)(1-e^2)$$

$$\therefore y^2=(ae-x)^2(1-e^2)^2$$

Substitute into (1)

$$\frac{x^2}{a^2}+\frac{(ae-x)^2(1-e^2)^2}{a^2(1-e^2)}=1$$

$$x^2+(ae-x)^2(1-e^2)=a^2$$

$$x^2+(a^2e^2-2aex+x^2)(1-e^2)=a^2$$

$$x^2+a^2e^2-2aex+x^2-a^2e^4+2ae^3x-e^2x^2-a^2=0$$

$$2x^2-e^2x^2-2aex+2ae^3x+a^2e^2-a^2e^4-a^2=0$$

$$(2-e^2)x^2-2ae(x-e^2x)+a^2(e^2-e^4-1)=0$$

$$(2-e^2)x^2-2aex(1-e^2)+a^2(e^2-e^4-1)=0$$

x_1 and x_2 are the roots of this equation as $P(x_1,y_1)$ and $Q(x_2,y_2)$ are the points of intersection of (1) and (2)

(iii)

$$x_M = \frac{x_1 + x_2}{2}$$

From (ii),
$$x_1 + x_2 = \frac{2ae\left(1-e^2\right)}{2-e^2}$$

$$\frac{x_1 + x_2}{2} = \frac{ae\left(1-e^2\right)}{2-e^2}$$

M lies on PQ: Sub in $M\left(x_M, y_M\right)$

$$\frac{ae\left(1-e^2\right)}{2-e^2} \times \frac{1}{ae} + \frac{y_M}{ae\left(1-e^2\right)} = 1$$

$$\frac{1-e^2}{2-e^2} + \frac{y_M}{ae\left(1-e^2\right)} = 1$$

$$\frac{y_M}{ae\left(1-e^2\right)} = 1 - \frac{1-e^2}{2-e^2}$$

$$\frac{y_M}{ae\left(1-e^2\right)} = \frac{2-e^2-1+e^2}{2-e^2}$$

$$\frac{y_M}{ae\left(1-e^2\right)} = \frac{1}{2-e^2}$$

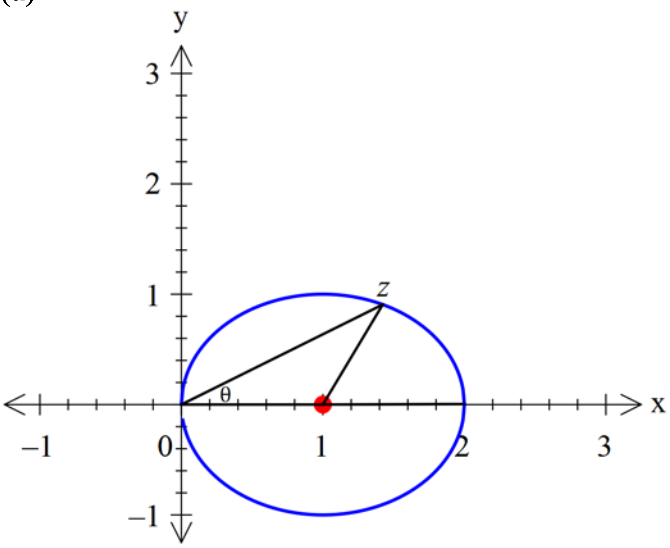
$$y_M = \frac{ae\left(1-e^2\right)}{2-e^2}$$

$$= x_M$$

$\therefore M$ lies on the line $y = x$.

Question 16

(a)



$\arg(z) = \theta$

$z-1$ is the vector from 1 to z

Let $\arg(z-1) = \beta$

$2\theta = \beta$ (angle at the centre is twice the angle at the circumference, subtended by the same arc)

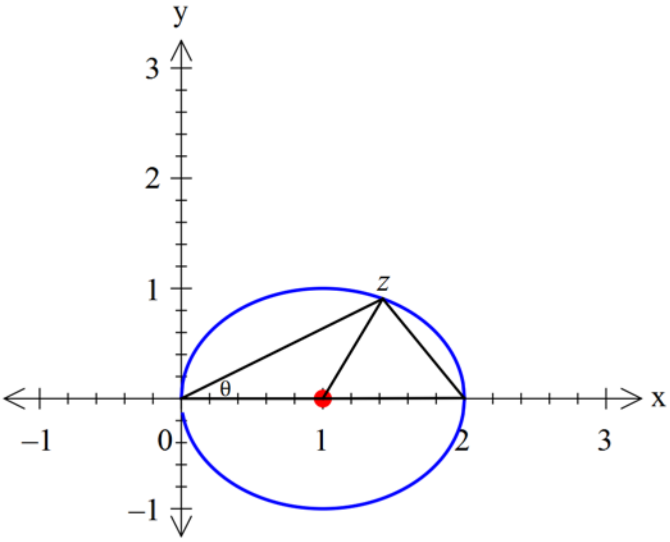
$\arg(z-1) = 2\theta$

(ii)

$\arg(z^2 - 3z + 2) = \arg[(z-2)(z-1)]$

$= \arg(z-2) + \arg(z-1)$

$z-2$ is the vector from 2 to z



The angle between the vector z and $z-2$ is $\frac{\pi}{2}$ (angle in semicircle)

$\arg(z-2) = \frac{\pi}{2} + \theta$ (exterior angle of a triangle)

$$\arg(z^2 - 3z + 2) = \frac{\pi}{2} + \theta + 2\theta$$

$$= \frac{\pi}{2} + 3\theta$$

(b)

(i)

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{a}{a}+\frac{a}{b}+\frac{b}{a}+\frac{b}{b}$$

$$=2+\frac{a}{b}+\frac{b}{a}$$

But

$$a+\frac{1}{a}\geq 2$$

$$\therefore \frac{a}{b}+\frac{b}{a}\geq 2$$

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)=2+\frac{a}{b}+\frac{b}{a}$$

$$\geq 4$$

(ii)

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4$$

$$(b+c)\left(\frac{1}{b}+\frac{1}{c}\right)\geq 4$$

$$(c+a)\left(\frac{1}{a}+\frac{1}{c}\right)\geq 4$$

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)(b+c)\left(\frac{1}{b}+\frac{1}{c}\right)(c+a)\left(\frac{1}{a}+\frac{1}{c}\right)\geq 64$$

$$(a+b)(b+c)(c+a)\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{1}{c}\right)\geq 64$$

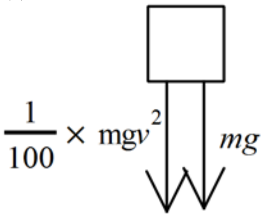
$$(a+b)(b+c)(c+a)\left(\frac{b+a}{ab}\right)\left(\frac{c+b}{bc}\right)\left(\frac{c+a}{ac}\right)\geq 64$$

$$(a+b)^2(b+c)^2(c+a)^2\geq 64ab\times bc\times ac$$

$$\left[(a+b)(b+c)(c+a)\right]^2\geq (8abc)^2$$

$$\therefore (a+b)(b+c)(c+a)\geq 8abc$$

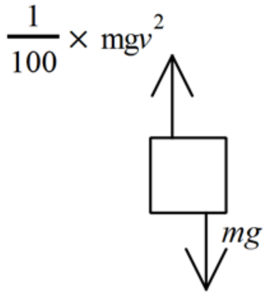
(c)
(i)



$$m\ddot{x} = \frac{-1}{100} mgv^2 - mg$$
$$\ddot{x} = \frac{-1}{100} gv^2 - g$$
$$= \frac{-1}{100} gv^2 - \frac{100g}{100}$$
$$= -\frac{1}{100} g\left(v^2 + 100\right)$$

(ii)
$$v.\frac{dv}{dx} = \frac{-1}{100} g\left(v^2 + 100\right)$$
$$\frac{dv}{dx} = \frac{-1}{100} g\left(\frac{v^2 + 100}{v}\right)$$
$$\frac{dx}{dv} = -100g\left(\frac{v}{v^2 + 100}\right)$$
$$x = -50g\int \frac{2v}{v^2 + 100} dv$$
$$= -50g \ln\left(v^2 + 100\right) + C$$
When $x = 0, v = u$
$$0 = \frac{-50}{g} \ln\left(u^2 + 100\right) + c$$
$$c = \frac{50}{g} \ln\left(u^2 + 100\right)$$
$$x = \frac{-50}{g} \ln\left(v^2 + 100\right) + \frac{50}{g}\left(u^2 + 100\right)$$
Let $v = 0$
$$x = -\frac{50}{g} \ln 100 + \frac{50}{g} \ln\left(u^2 + 100\right)$$
$$= 50g\left(\ln\left(u^2 + 100\right) - \ln 100\right)$$
$$= 50g \ln\left(\frac{u^2 + 100}{100}\right)$$

(iii)



$$m\ddot{x} = mg - \frac{1}{100}mgv^2$$

$$\begin{aligned}\ddot{x} &= g - \frac{1}{100}gv^2 \\ &= \frac{100g}{100} - \frac{1}{100}gv^2 \\ &= \frac{1}{100}\left(100g - gv^2\right) \\ &= \frac{g}{100}\left(100 - v^2\right)\end{aligned}$$

(iv)

$$v\frac{dv}{dx} = \frac{1}{100}g\left(100 - v^2\right)$$

$$\frac{dv}{dx} = \frac{1}{100}g\left(\frac{100 - v^2}{v}\right)$$

$$\frac{dx}{dv} = \frac{100}{g}\left(\frac{v}{100 - v^2}\right)$$

$$\begin{aligned}x &= \frac{-50}{g}\int\frac{-2v}{100 - v^2}dv \\ &= \frac{-50}{g}\ln\left(100 - v^2\right) + C\end{aligned}$$

When $x = 0, v = 0$

$$0 = -\frac{50}{g}\ln 100 + c$$

$$c = \frac{50}{g}\ln 100$$

$$\begin{aligned}x &= -\frac{50}{g}\ln\left(100 - v^2\right) + \frac{50}{g}\ln 100 \\ &= \frac{50}{g}\ln\left(\frac{100}{100 - v^2}\right)\end{aligned}$$

Let $x = \frac{50}{g}\ln\left(\frac{100 + u^2}{100}\right)$

$$\frac{100+u^2}{100}=\frac{100}{100-v^2}$$

$$100-v^2=\frac{100^2}{100+u^2}$$

$$v^2=100-\frac{100^2}{100+u^2}$$

$$v^2=\frac{100^2+100u^2-100^2}{100+u^2}$$

$$v^2=\frac{100u^2}{100+u^2}$$

$$v=\frac{10u}{\sqrt{100+u^2}}\quad \sin ce\ u>0$$

(v)

$$Let\ \ddot{x}=0, V=v$$

$$0=\frac{1}{100}g\left(100-V^2\right)$$

$$100-V^2=0$$

$$V^2=100$$

$$V=10\quad (V>0)$$

(vi)

Let $u=V$

$$\frac{10V}{\sqrt{100+V^2}}=\frac{100}{\sqrt{100+100}}$$

$$=\frac{100}{\sqrt{200}}$$

$$=\frac{100}{10\sqrt{2}}$$

$$=\frac{10}{\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}}V$$